OPEN MAPPINGS AND DIMENSION

By

DOROTHY ALICE MASON

A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL
OF THE UNIVERSITY OF FLORIDA IN
PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

1983

ACKNOWLEDGMENTS

I would like to express my thanks and appreciation to my advisor, Dr. David C. Wilson, for his long-lasting help, patience and encouragement. His friendship and support have been invaluable to me during my career as a graduate student.

I would also like to thank Dr. John Walsh for the assistance and encouragement he has given me.

Finally, I would like to acknowledge the remaining members, both past and present, of my supervisory committee: Dr. Thomas E. Bullock, Dr. Charles A. Burnap, Dr. Robert R. Kallman, Dr. James E. Keesling, Dr. Joseph McKenna, and Dr. Mark L. Teply.

TABLE OF CONTENTS

page

ACKNOWLEDGEMENTS.. ii

ABSTRACT.. iv

CHAPTER I INTRODUCTION.............................. 1

1.1 Dimension-Raising and Dimension-Lowering Mappings.................................. 1
1.2 Completely Regular Maps.................... 5
1.3 Preliminary Definitions.................... 6

CHAPTER II A DIMENSION-PRESERVING OPEN MAPPING WITH 1-DIMENSIONAL ANR POINT-INVERSES............ 8

2.1 Some Notation.............................. 8
2.2 The Space W_n.............................. 9
2.3 The Inverse System $\{X_n, f_n\}$ 12
2.4 The Induction.............................. 15
2.5 The Dimension of X......................... 18
2.6 The Point-Inverses of f.................... 21

CHAPTER III MONOTONE COMPLETELY REGULAR MAPS........ 26

REFERENCES.. 30

BIOGRAPHICAL SKETCH................................ 32

Abstract of Dissertation Presented to the Graduate School
of the Univesity of Florida in Partial Fulfillment of the
Requirements for the Degree of Doctor of Philosophy

OPEN MAPPINGS AND DIMENSION

By

DOROTHY ALICE MASON

April 1983

Chairman: David C. Wilson
Major Department: Mathematics

This dissertation contains two main results. The first is an example of a 1-dimensional continuum X and an open map f of X onto the unit interval I such that for each $y \in Y$, $f^{-1}(y)$ is a 1-dimensional ANR. This example shows the necessity of the existence of ε in the following theorem of E. Dyer: If f is an open map of the compact metric space X onto the finite-dimensional metric space Y such that for each $y \in Y$, $f^{-1}(y)$ is a nondegenerate Peano continuum, and there exists a positive number ε such that if $y \in Y$ and $f^{-1}(y)$ contains a simple closed curve C then diam $C > \varepsilon$, then $\dim X = \dim Y + 1$.

The second result states that if f is a monotone completely regular map of a 1-dimensional continuum onto a nondegenerate continuum, then f is cell-like. This improves the conclusion of Roger's version of Jones' Aposyndetic Decomposition Theorem by showing that the elements of Rogers' decomposition are tree-like.

CHAPTER I

INTRODUCTION

1.1 Dimension-Raising and Dimension-Lowering Mappings

It has long been known that mappings (continuous functions) have no respect for dimension. For example, if $I = [0,1]$ is the unit interval, the projection $P_1: I \times I \to I$ of the product $I \times I$ onto its first factor, defined by $P_1(x_1,x_2) = x_1$, takes a 2-dimensional space onto a 1-dimensional space. On the other hand, there is the famous mapping, due originally to Peano, of I onto $I \times I$, thus sending a 1-dimensional space onto a space of dimension 2.

It is natural to ask, therefore, what kinds of conditions one might impose on a function in order to insure that it does, or does not, lower (or raise) dimension. This question has generated a considerable amount of research and some quite interesting results.

Hurewicz [7] and Mazurkiewicz [17] showed independently that any continuum is the image of some 1-dimensional continuum under a monotone mapping. Thus monotone mappings can either raise or lower dimension. The mappings of Hurewicz and Mazurkiewicz, however, are not open.

Kolmogoroff [11] and Kazdan [10] showed that there are open mappings which raise dimension, but their maps were not monotone. Since the projection map P_1 defined above is open and lowers dimension, we see that open mappings also have no respect for dimension.

In reference 1, Anderson announced that there is a monotone open map of the universal curve, which is a 1-dimensional locally connected continuum, onto any locally connected continuum such that each point-inverse is homeomorphic to the universal curve. However, he never published a proof. Wilson [23] published a proof in 1970. This result shows that even monotone open maps having 1-dimensional, locally connected point-inverses can raise or lower dimension.

A question of great current interest is the Cell-like Mapping Problem: Can a cell-like map raise dimension? At the time of this writing, it is known that if the answer is yes, then the range must be infinite-dimensional. There are several other partial results (see, for example, Kozlowski and Walsh [12]) but the general problem remains unsolved.

Let X and Y be two spaces. The projection map $P: X \times Y \to X$ defined by $P(x,y) = x$ is an open map. If $\dim X \times Y = \dim X + \dim Y$, and $\dim Y > 0$, then the map P lowers dimension. However, the following theorem, which can be found in Hurewicz and Wallman [8], shows that it may not necessarily be the case that $\dim X \times Y = \dim X + \dim Y$.

Theorem 1.1. For any two spaces X and Y, $\dim X \times Y \leq \dim X + \dim Y$, and the inequality may be strict.

In fact, Pontrjagin [18] showed that for each prime p there is a compact metric space Φ_p such that $\dim \Phi_p = 2$ and

$$\dim \Phi_p \times \Phi_{p'} = \begin{cases} 3 \text{ if } p \neq p' \\ 4 \text{ if } p = p' \end{cases} .$$

Later, Boltyanskii [2] found an example of a compact metric space X such that dim X = 2 and dim X × X = 3. A good discussion of both examples can be found in Williams [21]. For either example, projection onto one of the factors would still yield a dimension-lowering open map, and the situation is even more encouraging due to the following theorem of Fáry [5].

Theorem 1.2. If X is a locally compact metric space and $\dim X \geq n$, then $\dim X \times X \geq 2n-1$.

Thus, if $\dim X \geq 2$, then $\dim X \times X \geq 3$ and the projection of X × X onto X yields an open, dimension-lowering map. Also, Cohen [3] has shown that if dim X = 1, then dim X × X = 2. Thus, in all finite-dimensional cases, projection of X × X onto X yields an open dimension-lowering map.

Two theorems of Dyer [4] shed further light on the subject of dimension-lowering maps. The first is

Theorem 1.3. If f is a mapping of the compact metric space M onto the n-dimensional metric space N and for each

$y \in N$ and nonnegative integer k, $\tilde{H}_k(f^{-1}(y)) = 0$, and there is an integer j such that $\dim f^{-1}(y) \leq j$ for each $y \in N$, then $n \leq \dim M \leq n + j$.

Notice that the mapping in Theorem 1.3 is monotone but not necessarily open. By requiring that the map f be open, Dyer obtained the following stronger result.

Theorem 1.4. If f is an open mapping of the compact metric space M onto the n-dimension metric space N such that $f^{-1}(y)$ is a 1-dimensional ANR for each $y \in N$, and there exists $\varepsilon > 0$ such that for no $y \in Y$ does $f^{-1}(y)$ contain a simple closed curve of diameter less than ε, then $\dim M = n + 1$.

In Chapter 2 we give an example which shows that without the ε-condition in the above theorem, the function f might not lower dimension. This example is a 1-dimensional continuum X along with a monotone open map f of X onto I such that for each $y \in I$, $f^{-1}(y)$ is a 1-dimensional ANR. We thus have a 1-dimensional space and a mapping defined on it satisfying all the condition of Dyer's Theorem 1.4, except for the existence of the lower bound ε, and the map f preserves dimension, rather than lowering it.

Some immediate questions arise in relation to this example.

1. The space X in the example cannot be embedded in the plane. Thus one is led to ask if there is a planar continuum having the same properties.

2. Is the space X locally connected? If not, is there a locally connected space having the same properties?

3. Is there an example of a compact metric space X and a monotone open map f of X onto some space Y such that $f^{-1}(y)$ is a 1-dimensional ANR, and dim Y > dim X?

The author is grateful to Professor Lex Oversteegen for pointing out the following theorem of Krasinkewicz and Minc [13], which gives a negative answer to question one.

Theorem 1.5. If X is a 1-dimensional planar continuum and f is a monotone open map of X onto the nondegenerate continuum Y such that either

(1) $f^{-1}(y)$ is a λ-dendroid for each $y \in Y$ or

(2) $f^{-1}(y)$ is locally connected for each $y \in Y$, then f is a homeomorphism.

1.2 Completely Regular Maps

In Chapter 3 we prove the following theorem concerning completely regular maps.

Theorem 3.2. If f is a monotone completely regular map of a 1-dimensional continuum X onto a nondegenerate continuum Y, then f is cell-like.

Part of the conclusion of Corollary 3.3 is that dim Y = 1, so that the map f must preserve dimension.

The assumption that dim X = 1 is essential, because the example in reference 21 shows that there exist monotone completely regular maps which are not cell-like. The domain in this example is 2-dimensional.

In reference 16 it is shown that if A is an n-dimensional continuum, then there is an n-dimensional continuum X and a monotone open map f of X onto I such that for each $y \in Y$, $f^{-1}(y)$ has the shape of either A or A v A, the wedge of A with itself. By letting A be any 1-dimensional continuum with nontrivial shape, it is evident that the assumption of complete regularity in Theorem 3.2 is necessary.

Theorem 3.2 improves the conclusion of Roger's version [19] of Jones' Aposyndetic Decomposition Theorem [9]. For a complete statement of Rogers' Decomposition Theorem, the reader is referred to reference 19. For the purpose of this comment it is sufficient to note that Jones' theorem gives a continuous decomposition G of a certain type of 1-dimensional continuum M such that the quotient map $\pi: M \to M/G$ is monotone. Rogers showed that π is completely regular and acyclic. Theorem 3.2 shows that π is cell-like, and hence the elements of G are tree-like.

1.3 Preliminary Definitions

A compactum is a compact metric space. A continuum is a connected compactum. If ϕ and θ are two functions of a space X into a metric space Y with metric d, then $d(\phi,\theta) = \sup\{d(\phi(x),\theta(x)): x \in X\}$. If $\varepsilon > 0$ and $d(\phi,\theta) < \varepsilon$, we will say that ϕ and θ are ε-close.

The space X is an ANR, or absolute neighborhood retract, if whenever X is embedded as a closed subspace of a normal space N, there is an open set U in N which contains X and a map $r: U \to X$ such that $r(x) = x$ for each $x \in X$.

Let f be a map of a space X onto a space Y. The map f is open if f(U) is open in Y for each open set $U \subset X$. The map f is monotone provided that for each $y \in Y$, $f^{-1}(y)$ is connected. The map f is completely regular provided that for each $y \in Y$ and $\varepsilon > 0$ there exists an open set $V \subset Y$ with $y \in V$ such that if $z \in V$ then there is a homeomorphism h of $f^{-1}(y)$ onto $f^{-1}(z)$ such that $d(x,h(x)) < \varepsilon$ for each $x \in f^{-1}(y)$. The map f is cell-like (or CE) provided that, for each $y \in Y$, $f^{-1}(y)$ is cell-like. (A finite dimensional compactum Z is cell-like if for any map $f: Z \to P$, where P is a polyhedron, f is inessential; see Lacher [14, 15] for more details on cell-like spaces and cell-like mappings.)

A subset A of a space X is contractible in X if there is a homotopy $H: A \times I \to X$ such that H_0 is the identity and $H_1(A)$ is a point.

CHAPTER II

A DIMENSION-PRESERVING OPEN MAPPING WITH 1-DIMENSIONAL ANR POINT-INVERSES

In this chapter we construct a compact 1-dimensional space X and an open map f of X onto the unit interval I such that, for each $y \in I$, $f^{-1}(y)$ is either an arc or an "eyebolt."

2.1 Some Notation

For i = 1,2,3 define $P_i \colon E^3 \to \mathbb{R}$ by $P_i(x_1,x_2,x_3) = x_i$. Also, define $P \colon E^3 \to E^2$ by $P(x,y,z) = (x,y,0)$.

It will be convenient later on to have ε-neighborhoods be cubes and so we will use for the metric on E^3 the function

$d((x_1,x_2,x_3),\ (x_1',x_2',x_3')) = \max\{|x_i-x_i'|: i = 1,2,3\}$.

The space X will be the inverse limit of an inverse system $\{X_n, f_n\}$, where each X_n is a two-dimensional subspace of E^3. For this space, we will use the metric

$$\rho((u_n)_{n=1}^{\infty},\ (v_n)_{n=1}^{\infty}) = \sup\{\tfrac{1}{n}\, d(u_n,v_n): n = 1,2,\ldots\}.$$

Let A_0, A_1, A_2, ... be a listing of the rational numbers in I = [0,1], with $A_0 = 0$ and $A_1 = 1$. For $n \geq 2$, let $\ell(n)$ and $r(n)$ be integers such that

$A_{\ell(n)} = \max\{A_i: A_i < A_n, i < n\}$ and $A_{r(n)} = \min\{A_i: A_i > A_n, i < n\}$. Note that $A_{\ell(n)}$ is the largest rational which precedes A_n in the list and is "to the left" of A_n. Similarly, $A_{r(n)}$ is the smallest rational which precedes A_n in the list and is "to the right" of A_n.

Let $X_1 = I \times I \times \{0\} \subset E^3$. For $n \geq 2$, the space X_n will be constructed from X_{n-1} by replacing $\{(x,y,z) \in X_{n-1}: A_{\ell(n)} \leq x \leq A_{r(n)}\}$ with a new space W_n.

2.2 The Space W_n

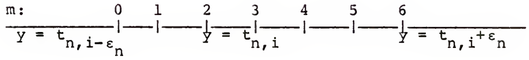
Assume that for $n \geq 2$ we have chosen a certain number $k(n)$ of distinct points $t_{n,1}, t_{n,2}, \ldots, t_{n,k(n)}$ in I and a number $\varepsilon_n > 0$ such that the intervals $I_{n,i} = [t_{n,i} - \varepsilon_n , t_{n,i} + \varepsilon_n]$ are pairwise disjoint and contained in the interior of I. Assume also that we have chosen two irrational numbers p_n and q_n such that $A_{\ell(n)} < p_n < A_n < q_n < A_{r(n)}$. These choices are crucial, and the procedure for making them will be described inductively later in this chapter.

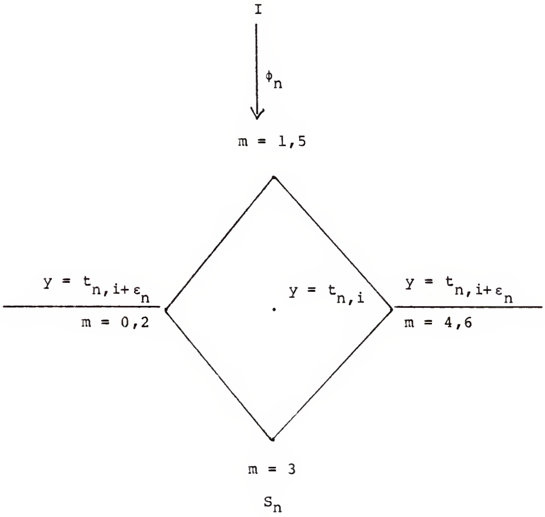
Let $S_0 = \{0\} \times I \times \{0\}$ and $S_1 = \{1\} \times I \times \{0\}$. For $n \geq 2$, let S_n be the space obtained from $\{A_n\} \times I \times \{0\}$ by replacing each subinterval $\{A_n\} \times I_{n,i} \times \{0\}$ with the square having $I_{n,i}$ as one of its diagonals.

Define $\phi_0: I \to S_0$ by $\phi_0(t) = (0,t,0)$, and $\phi_1: I \to S_1$ by $\phi_1(t) = (1,t,0)$. For $n \geq 2$ define $\phi_n: I \to S_n$ by defining $\phi_n(0) = (A_n,0,0)$, $\phi_n(1) = (A_n,1,0)$ and, for i = 1,2,...,k(n) and m = 0,1,2,...,6,

$$\phi_n\left(t_{n,i} - \varepsilon_n + \frac{m\varepsilon_n}{3}\right) = \begin{cases} (A_n, t_{n,i} - \varepsilon_n, 0) & \text{for } m = 0,2 \\ (A_n, t_{n,i}, \varepsilon_n) & \text{for } m = 1,5 \\ (A_n, t_{n,i}, -\varepsilon_n) & \text{for } m = 3 \\ (A_n, t_{n,i} + \varepsilon_n, 0) & \text{for } m = 4,6 \end{cases}$$

and extending linearly.





For each $n \geq 0$ define $\Theta_n: I \to S_n$ by $\Theta_n(t) = (x,y,-z)$, where $(x,y,z) = \phi_n(t)$.

The fact that Θ_n is the reflection of ϕ_n will play an important role in ensuring the 1-dimensionality of X.

Now, define $L_n: \{(x,y,0) \in X_1: A_{\ell(n)} \leq x \leq A_n\} \to E^3$ by defining

$$L_n(x,y,0) = \begin{cases} \phi_n(y) & \text{if } x = A_n \\ \Theta_{\ell(n)}(y) & \text{if } x = A_{\ell(n)} \\ (x,y,0) & \text{if } x = p_n \end{cases}$$

and extending linearly. Also, define $R_n: \{(x,y,0) \in X_1: A_n \leq x \leq A_{r(n)}\} \to E^3$ by defining

$$R_n(x,y,0) = \begin{cases} \Theta_n(y) & \text{if } x = A_n \\ \phi_{r(n)}(y) & \text{if } x = A_{r(n)} \\ (x,y,0) & \text{if } x = q_n \end{cases}$$

and extending linearly.

There are several important observations to make concerning the maps L_n and R_n.

1) $P_1 \circ L_n(x,y,0) = x$ and $P_1 \circ R_n(x,y,0) = x$.

2) L_n restricted to $\{(x,y,0): A_{\ell(n)} < x < A_n\}$ is a homeomorphism, as is R_n restricted to $\{(x,y,0): A_n < x < A_{r(n)}\}$.

3) If $(P \circ L_n)^{-1}(x,y,0)$ contains infinitely many points then either $x = \frac{1}{4}(A_n + 3p_n)$ or $x = \frac{1}{4}(A_{\ell(n)} + 3p_n)$.

Also, if $(P \circ R_n)^{-1}(x,y,0)$ contains infinitely many points, then either $x = \frac{1}{4}(A_n + 3q_n)$ or $x = \frac{1}{4}(A_{r(n)} + 3q_n)$.

Observation 3 is a straightforward exercise in basic analytic geometry.

We now let $W_n = \text{Im } L_n \cup \text{Im } R_n$. Note that W_n is actually the union of four mapping cylinders, one each for ϕ_n, θ_n, $\theta_{\ell(n)}$ and $\phi_{r(n)}$.

2.3 The Inverse System $\{X_n, f_n\}$

As above, let $X_1 = I \times I \times \{0\}$ and inductively, let $X_n = (X_{n-1} - \{(x,y,z): A_{\ell(n)} \leq x \leq A_{r(n)}\}) \cup W_n$.

Define $H_1: X_1 \to X_1$ to be the identity map. Define $H^2: X_1 - (\{A_2\} \times I \times \{0\}) \to X_2 - \{(x,y,z): x = A_2\}$ by

$$H_2(x,y,0) = \begin{cases} L_2(x,y,0) & \text{if } x < A_2 \\ R_2(x,y,0) & \text{if } x > A_2 \end{cases} .$$

Inductively, define

$$H_n: X_1 - (\{A_2, A_3, \ldots, A_n\} \times I \times \{0\}) \to$$
$$X_n - \{(x,y,z): x \in \{A_2, A_3, \ldots, A_n\}\}$$

by

$$H_n(x,y,0) = \begin{cases} L_n(x,y,0) & \text{if } A_{\ell(n)} \leq x < A_n \\ R_n(x,y,0) & \text{if } A_n < x \leq A_{r(n)} \\ H_{n-1}(x,y,0) & \text{otherwise.} \end{cases}$$

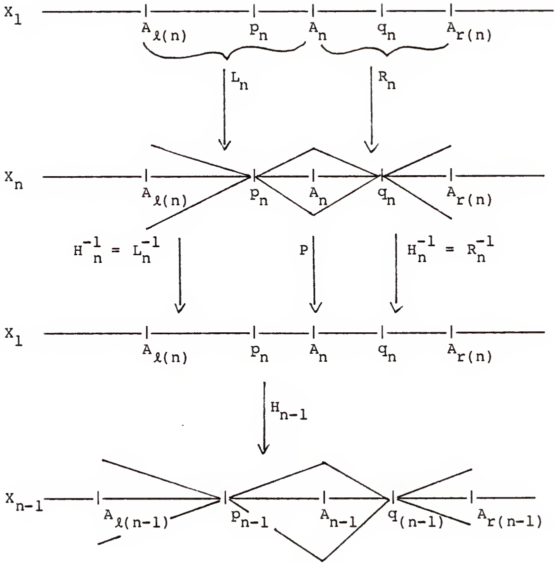
We have included in this definition the possibility that $x = A_{\ell(n)}$ or $x = A_{r(n)}$ because it may be the case that

$A_{\ell(n)} = 0$ or $A_{r(n)} = 1$. However, in any other case the inequalities should be taken to be strict.

Now define $f_n\colon X_n \to X_{n-1}$ by

$$f_n(x,y,z) = \begin{cases} H_{n-1}(x,y,0) & \text{if } p_n \le x \le q_n \\ H_{n-1} \circ H_n^{-1}(x,y,z) & \text{if } A_{\ell(n)} < x < p_n \\ & \text{or } q_n < x < A_{r(n)} \\ (x,y,z) & \text{otherwise.} \end{cases}$$

A closer examination of the map f_n now seems appropriate. The open interval $(A_{\ell(n)}, A_{r(n)})$ contains none of the rationals which appear in our list before A_n. Consequently H_{n-1} maps $(A_{\ell(n)}, A_{r(n)}) \times I \times \{0\}$ homeomorphically onto $\{(x,y,z) \in X_{n-1}\colon A_{\ell(n)} < x < A_{r(n)}\}$, which is the only portion of X_{n-1} which is changed in constructing X_n. We use this map to get from X_n to X_{n-1} by first "flattening" $\{(x,y,z) \in X_n\colon A_{\ell(n)} < x < A_{r(n)}\}$ into a subset of X_1. Since $H_{n-1}(x,y,0) = H_{n-1} \circ P(x,y,z)$, f_n is the projection into X_1 followed by H_{n-1} for $p_n \le x \le q_n$. For $x \in (A_{\ell(n)}, p_n) \cup (q_n, A_{r(n)})$, f_n first "undoes" H_n and then applies H_{n-1} to the resulting subset of X_1. Schematically, we have the following diagram, which indicates the action of f_n in the "X-direction" for $y = t_{n,i}$ for some i.



Observe that the following diagram commutes, for each n.

$$\begin{tikzcd}
X_{n-1} \arrow[d, "P_1|X_{n-1}"] & X_n \arrow[l, "f_n"'] \arrow[d, "P_1|X_n"] \\
I & I \arrow[l, "1_I"]
\end{tikzcd}$$

Thus, letting $X = \varprojlim\{X_n, f_n\}$ the open maps $P_1|_{X_n}$ induce an open map f of X onto I. Since for $m > n$ $f_m|_{S_n}$ is the identity map, $f^{-1}(A_n)$ is homeomorphic to S_n, for each n. If, on the other hand, t is an irrational number in I, $f^{-1}(t)$ is an inverse limit of arcs, which might not be an arc. However, the very careful choices made in the induction below will insure that $f^{-1}(t)$ is an arc, and that $\dim X = 1$.

2.4 The Induction

For $n = 2$, let β_2 be a positive irrational number such that $2\beta_2 < \frac{1}{4}$. Let C_2 be a chain of open sets in X_1 from $(A_2,0,0)$ to $(A_2,1,0)$ covering $\{A_2\} \times I \times \{0\}$ such that each link of C_2 is a β_2-neighborhood of some point $(A_2,y,0)$. (Recall that the metric we are using causes β_2-neighborhoods to be squares.)

Choose the number k(2) and the points $t_{2,1}, t_{2,2}, \ldots, t_{2,k(2)}$ so that each link of C_2 contains at least one point of the form $(A_2, t_{2,i}, 0)$, and so that $t_{2,i} = \frac{1}{2}$ for some i.

Choose the two irrationals p_2 and q_2 so that $A_2 - \beta_2 < p_2 < A_2 < q_2 < A_2 + \beta_2$. The chain C_2 is an open cover of the compact space $[p_2,q_2] \times I \times \{0\}$. Let λ_2 be a Lebesgue number for this cover and choose $\varepsilon_2 > 0$ so that $\varepsilon_2 < \frac{1}{2}\lambda_2$.

Using these values, construct the spaces W_2 and X_2 and the maps f_2 and H_2 as described above.

Now, assume that $n > 2$ and that for each $j < n$ we have made the choices necessary for the construction of W_j and X_j

and the definition of the maps H_j and f_j. Further assume that for each $j < n$ each of the following six conditions is satisfied.

1) There is a number $\gamma_j > 0$ such that if $u,v \in \{(x,y,0) \in X_1 : A_{\ell(n)} \leq x \leq A_{r(n)}\}$ with $d(u,v) < \gamma_j$, then

$$d(f_{j-1,\ell} \circ \hat{H}_{j-1}(u),\ f_{j-1,\ell} \circ \hat{H}_{j-1}(v)) < \min\{\varepsilon_{j-1}, \frac{1}{2^j}\}$$

for all $\ell \leq j-1$, where $\hat{H}_{j-1}$ is the "extension" of H_{j-1} to the compact space $[A_{\ell(j)}, A_{r(j)}] \times I \times \{0\}$ defined by

$$\hat{H}_{j-1}(x,y,0) = \begin{cases} H_j(x,y,0) & \text{if } x \notin \{A_{\ell(j)}, A_{r(j)}\} \\ \Theta_{\ell(j)}(Y) & \text{if } x = A_{\ell(j)} \\ \phi_{r(j)}(Y) & \text{if } x = A_{r(j)} \end{cases} .$$

2) There is a chain C_j of open sets in X_1 from $(A_j,0,0)$ to $(A_j,1,0)$ covering $\{A_j\} \times I \times \{0\}$ with Lebesgue number λ_j such that each link of C_j is a β_j-neighborhood of some point $(A_j,y,0)$, where $\beta_j > 0$ is chosen so that $\beta_j < \frac{1}{2}\min\{\gamma_j, \frac{1}{2^j}, \lambda_{j-1}\}$, β_j is irrational and

$$A_{\ell(j)} < A_j - \beta_j < A_j < A_j + \beta_j < A_{r(j)} .$$

3) The two irrationals p_j and q_j are chosen so that $A_j - \beta_j < p_j < A_j < q_j < A_j + \beta_j$.

4) The number $k(j)$ and the points $t_{j,1},\dots,t_{j,k(j)}$ are such that

(a) each link of C_j contains at least one point of the form $(A_j,t_{j,i},0)$, and

(b) if for some $\ell < j$ and $k = 1,2,\dots,j-1$, $f_{j-1,\ell} \circ H_{j-1}(A_j,y,0) = H_\ell(A_j, \frac{k}{j}, 0)$, then $y = t_{j,i}$ for some $i \in \{1,2,\dots,k(j)\}$.

5) The number ε_j is less than $\min\{\frac{1}{2} \lambda_j, \varepsilon_{j-1}\}$.

6) For any $\ell < j$ and $m = 1,2,\dots,k(\ell)$,

(a) if $t_{\ell,m} - \varepsilon_\ell < t_{j,i} < t_{\ell,m} + \varepsilon_\ell$, then $t_{\ell,m} - \varepsilon_\ell < t_{j,i} - \varepsilon_j < t_{j,i} + \varepsilon_j < t_{\ell,m} + \varepsilon_\ell$, and

(b) if $t_{j,i} \notin [t_{\ell,m} - \varepsilon_\ell, t_{\ell,m} + \varepsilon_\ell]$ then $[t_{j,i} - \varepsilon_j, t_{j,i} + \varepsilon_j] \cap [t_{\ell,m} - \varepsilon_\ell, t_{\ell,m} + \varepsilon_\ell] = \emptyset$.

The first step at the n^{th} stage is to "extend" the map H_{n-1} to the compact space $[A_{\ell(n)}, A_{r(n)}] \times I \times \{0\}$ by defining

$$\hat{H}_{n-1}(x,y,0) = \begin{cases} H_{n-1}(x,y,0) & \text{if } x \notin \{A_{\ell(n)}, A_{r(n)}\} \\ \Theta_{\ell(n)}(y) & \text{if } x = A_{\ell(n)} \\ \phi_{r(n)}(y) & \text{if } x = A_{\ell(n)} \end{cases} .$$

Since $\hat{H}_{n-1}$ is uniformly continuous, there is a number $\gamma_n > 0$ such that if $u,v \in \{(x,y,0) \in X_1 : A_{\ell(n)} \le x \le A_{r(n)}\}$ with $d(u,v) < \gamma_n$, then for any $\ell \le n-1$, $d(f_{n-1,\ell} \circ \hat{H}_{n-1}(u), f_{n-1,\ell} \circ \hat{H}_{n-1}(v)) < \varepsilon_{n-1}$.

Choose an irrational number $\beta_n > 0$ such that

$$\beta_n < \frac{1}{2}\min\{\gamma_n, \frac{1}{2^n}, \lambda_{n-1}\} \text{ and } A_{\ell(n)} < A_n - \beta_n < A_n + \beta_n < A_{r(n)}.$$

Let C_n be a chain of open sets in X_1 from $(A_n,0,0)$ to $(A_n,1,0)$ covering $\{A_n\} \times I \times \{0\}$, such that each link of C_n is a β_n-neighborhood of some point $(A_n,y,0)$. Choose the two irrational numbers p_n and q_n so that $A_n - \beta_n < p_n < A_n < q_n < A_n + \beta_n$. Let λ_n be a Lebesgue number for C_n, considered as an open cover of the compact space $[p_n,q_n] \times I \times \{0\}$.

We observed earlier that if $j < n$ and $(P \circ H_j)^{-1}(x,y,0)$ contains infinitely many points, then x must be irrational. Thus, the set $\{y\colon f_{n-1,j} \circ H_{n-1}(A_n,y,0) = H_j(A_n, \frac{m}{n}, 0)$ for some $j < n$ and $m = 1,2,\ldots,n-1\}$ is finite. Choose the number $k(n)$ and the points $t_{n,1},t_{n,2},\ldots,t_{n,k(n)}$ to include all the points in this set and so that each link of C_n contains at least one point of the form $(A_n,t_{n,i},0)$.

Choose $\varepsilon_n > 0$ such that $\varepsilon_n < \min\{\frac{1}{2}\lambda_n,\varepsilon_{n-1}\}$ and so that Property 6 holds for $j = n$.

This completes the inductive description of the choices necessary for the construction of the spaces W_n and X_n and the maps H_n and f_n.

2.5 The Dimension of X

Clearly, X is compact and $\dim X \geq 1$. In order to show that $\dim X \leq 1$, we will use the following theorem, whose proof can be found in reference 20.

Theorem 2.1. Let $\{P_q, f_q\}$ be an inverse sequence of compact polyhedra with a compactum X as its limit. The compactum X has dimension $\leq n$ if and only if for each k and each $\varepsilon > 0$ there is an integer $j > k$, a triangulation L_k of P_k and a map $g_{j,k}: P_j \to L_k^{(n)}$ which is ε-close to $f_{j,k}$.

Let the integer k and the number $\varepsilon > 0$ be given. Let G be an open cover of X_k with $\mu(G) < \varepsilon/3$, and let λ be a Lebesgue number for G. Let L_k be a triangulation of X_k with mesh less than λ. Find $j > k$ such that each triangle of L_k contains a point of the form $H_k(A_n, \frac{m}{n}, 0)$ for some n and m where $k < n \leq j$ and $m \in \{1,2,\ldots,n-1\}$. Choose one such point in each triangle of L_k and denote these points by $\omega_1, \omega_2, \ldots, \omega_\ell$.

Recall that the points $t_{n,1}, t_{n,2}, \ldots, t_{n,k(n)}$ were chosen to include the set

$$\{y\colon f_{n-1,j} \circ H_{n-1}(A_n, y, 0) = H_k(A_n, \frac{m}{n}, 0),\ m = 1,2,\ldots,n-1\}.$$

For any $i > n$, $f_i(A_n, y, z) = (A_n, y, z)$, and $f_n(A_n, y, z) = H_{n-1}(A_n, y, 0)$. Thus $f_{j,k}(A_n, y, z) = f_{n-1,k} \circ H_{n-1}(A_n, y, 0)$ and so

$$f_{j,k}^{-1}(\{H_k(A_n, \frac{m}{n}, 0)\colon 1 \leq m \leq n-1\})$$
$$\subset \{(A_n, t_{n,i}, z) \in X_j\colon 1 \leq i \leq k(n)\}$$

Thus, if $\phi\colon X_j \to X_j$ is a small "push" of X_j off of all the points $(A_n, t_{n,i}, z)$ where $k < n \leq j$ and $i = 1,2,\ldots,k(n)$, then $f_{j,k} \circ \psi$ will miss one point in each triangle of L_k. Then by radially retracting each triangle from its missed point onto its boundary, we obtain a map of X_j into the one-skeleton of L_k.

Let $\delta > 0$ be such that if $u,v \in X_j$ with $d(u,v) < \delta$, then $d(f_{j,k}(u),f_{j,k}(v)) < \lambda$, and such that $\delta < \min\{d(A_i,A_m): i \le j,\ m \le j\}$.

For each $s = 1,2,\ldots,\ell$, $f^{-1}_{j,k}(\omega_s) = \{(A_n,t_{n,i},\varepsilon_n), (A_n,t_{n,i},-\varepsilon_n)\}$ for some n and i with $k < n \le j$. Because of the construction of X_j, in particular the fact that

$(A_n,t_{n,i},\varepsilon_n) = \Theta_n(t_{n,i}) = \phi_n(t_{n,i} \pm \frac{2\varepsilon_n}{3})$ and

$(A_n,t_{n,i},-\varepsilon_n) = \phi_n(t_{n,i}) = \Theta_n(t_{n,i} \pm \frac{2\varepsilon_n}{3})$, the

δ-neighborhood in X_j of $(A_n,t_{n,i},\varepsilon_n)$ is homeomorphic to $[0,1) \times (0,1)$, with $(A_n,t_{n,i},\varepsilon_n)$ corresponding to the point $(0,\frac{1}{2})$. The same is true for the δ-neighborhood of $(A_n,t_{n,i},-\varepsilon_n)$. Thus there is a map $\psi: X_j \to X_j$ such that $d(u,\psi(u)) < \delta$ for any $u \in X_j$ and $f^{-1}_{j,k}(\{\omega_1,\omega_2,\ldots,\omega_\ell\}) \cap \operatorname{Im}\psi = \emptyset$.

Let $R: X_k - \{\omega_1,\omega_2,\ldots,\omega_\ell\} \to L_k^{(1)}$ be the map which radially retracts the triangle containing ω_s onto its boundary. Define $g_{j,k}: X_j \to L_k^{(1)}$ by $g_{j,k} = R \circ f_{j,k} \circ \psi$.

If $u \in X_j$, then $d(f_{j,k}(u), f_{j,k} \circ \psi(u)) < \lambda < \varepsilon/3$. Since each triangle of L_k has diameter less than $\varepsilon/3$, $d(v,R(v)) < \varepsilon/3$ for any $v \in X_j - \{\omega_1,\omega_2,\ldots,\omega_s\}$. Thus,

$$d(f_{j,k}(u), g_{j,k}(u)) \le d(f_{j,k}(u),f_{j,k}\circ \psi(u)) + d(f_{j,k} \circ \psi(u), g_{j,k}(u)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}\ .$$

Hence, $g_{j,k}$ and $f_{j,k}$ are ε-close, and we may conclude that dim X = 1.

2.6 The Point-Inverses of f

As noted earlier, for each n, $f^{-1}(A_n) = S_n$. We next show that all the other point-inverses of f are arcs.

Let $x_0 \in I$ be an irrational number. For each n, let $B_n = (P_1|_{X_n})^{-1}(x_0)$, and let $f_n^{x_0} = f_n|_{B_n}$. Then $f^{-1}(x_0) = \varprojlim\{B_n, f_n^{x_0}\}$, and each B_n is an arc from $(x_0,0,0)$ to $(x_0,1,0)$.

Let $\pi_n\colon X \to X_n$ be the natural projection of X onto X_n. There are two points a and b in X such that $\{a\} = \pi_n^{-1}(x_0,0,0)$ and $\{b\} = \pi_n^{-1}(x_0,1,0)$ for every n. We will show that $f^{-1}(x_0)$ is an arc by producing a sequence of chains from a to b covering $f^{-1}(x_0)$, with mesh approaching 0 such that each chain is "straight" in the one which precedes it.

Let $\{n(i)\}_{i=1}^{\infty}$ be the subsequence of the sequence of positive integers consisting of all integers n(i) such that $p_{n(i)} < x_0 < q_{n(i)}$. Before we define our chains, we make a useful computation to show that if $(x_0,y,z) \in X_{n(i+1)}$ then $f_{n(i+1),n(i)}(x_0,y,z) = H_{n(i)}(x_0,y,0)$, so that, letting $P^{x_0} = P|_{P_1^{-1}(x_0)}$, we have $f^{-1}(x_0)$ $= \varprojlim\{B_{n(i)}, H_{n(i-1)} \circ P^{x_0}\}$.

If n(i+1) = n(i) + 1, this follows directly from the definition of $f_{n(i+1)}$. Also, if there is no integer m between $n(i)$ and $n(i+1)$ such that $A_{\ell(m)} < x_0 < p_m$ or $q_m < x_0 < A_{r(m)}$, then $f_k^{x_0}$ is the identity map for each $k \in \{n(i)+1,\ldots, n(i+1)-1\}$. In this case we have, since $x_0 \notin [A_{\ell(m)},A_{r(m)}]$,

$$f_{n(i+1),n(i)}(x_0,y,z) = f_{n(i+1)-1,n(i)} \circ H_{n(i+1)-1}(x_0,y,0)$$

$$= H_{n(i+1)-1}(x_0,y,0) = H_{n(i+1)-2}(x_0,y,0) = \ldots$$

$$= H_{n(i)+1}(x_0,y,0) = H_{n(i)}(x_0,y,0).$$

Otherwise, let $m(1)$, $m(2)$, ..., $m(\ell)$ be all the integers such that $n(i+1) > m(1) > m(2) > \ldots > m(\ell) > n(i)$ and $A_{\ell(m(s))} < x_0 < p_{m(s)}$ or $q_{m(s)} < x_0 < A_{r(m(s))}$. Since if k is an integer between $n(i)$ and $n(i+1)$ and $k \notin \{m(s)\colon s = 1,2,\ldots,\ell\}$, $f_k^{x_0}$ is the identity map, and $H_k(x_0,y,0) = H_{k-1}(x_0,y,0)$, we have

$$\begin{aligned} f_{n(i+1),n(i)}(x_0,y,z) &= f_{n(i+1)-1,n(i)} \circ H_{n(i+1)-1}(x_0,y,0) \\ &= f_{n(i+1)-1,n(i)} \circ H_{m(1)}(x_0,y,0) \\ &= f_{m(1),n(i)} \circ f_{n(i+1)-1,m(1)} \circ H_{m(1)}(x_0,y,0) \\ &= f_{m(1),n(i)} \circ H_{m(1)}(x_0,y,0) \\ &= f_{m(1)-1,n(i)} \circ f_{m(1)} \circ H_{m(1)}(x_0,y,0) \\ &= f_{m(1)-1,n(i)} \circ H_{m(1)-1} \circ H_{m(1)}^{-1} \circ H_{m(1)}(x_0,y,0) \end{aligned}$$

$$= f_{m(1)-1,n(i)} \circ H_{m(1)-1}(x_0,y,0)$$

$$= f_{m(1)-1,n(i)} \circ H_{m(2)}(x_0,y,0)$$

$$= f_{m(2)-1,n(i)} \circ f_{m(2)} \circ f_{m(1)-1,m(2)} \circ H_{m(2)}(x_0,y,0)$$

$$= f_{m(2)-1,n(i)} \circ f_{m(2)} \circ H_{m(2)}(x_0,y,0)$$

$$= f_{m(2)-1,n(i)} \circ H_{m(2)-1} \circ H^{-1}_{m(2)} \circ H_{m(2)}(x_0,y,0)$$

$$= f_{m(2)-1,n(i)} \circ H_{m(2)-1}(x_0,y,0)$$

$$= f_{m(2)-1,n(i)} \circ H_{m(3)}(x_0,y,0).$$

Continuing in this manner, we obtain

$$f_{n(i+1),n(i)}(x_0,y,0) = f_{m(\ell)-1,n(i)} \circ H_{m(\ell)-1}(x_0,y,0)$$

$$= f_{m(\ell)-1,n(i)} \circ H_{n(i)}(x_0,y,0) = H_{n(i)}(x_0,y,0).$$

Recall that, in the inductive hypothesis, there is for each i a chain $\mathcal{C}_{n(i)}$ of open sets from $(A_{n(i)},0,0)$ to $(A_{n(i)},1,0)$ which covers $[p_{n(i)},q_{n(i)}] \times I \times \{0\}$. For $C^{n(i)} \in \mathcal{C}_{n(i)}$, let $\hat{C}^{n(i)}$ denote the open subset of $B_{n(i)}$ given by $B_{n(i)} \cap C^{n(i)}$. Let $\mathcal{V}_{n(i)} = \{H_{n(i)}(\hat{C}^{n(i)}): C^{n(i)} \in \mathcal{C}_{n(i)}\}$ and let $\mathcal{U}_{n(i)} = \{\pi^{-1}_{n(i)}(H_{n(i)}(\hat{C}^{n(i)})): C^{n(i)} \in \mathcal{C}_{n(i)}\}$. The chain $\mathcal{V}_{n(i)}$ covers the arc $B_{n(i)}$, and $\{\mathcal{U}_{n(i)}\}_{i=1}^{\infty}$ is a sequence of chains covering $f^{-1}(x_0)$. We will show that it is the desired sequence.

We must first show that $\mathcal{U}_{n(i+1)}$ refines $\mathcal{U}_{n(i)}$. Let $U^{n(i+1)} \in \mathcal{U}_{n(i+1)}$ with $U^{n(i+1)} = \pi^{-1}_{n(i+1)}(V^{n(i+1)})$, where $V^{n(i+1)} = H_{n(i+1)}(\hat{C}^{n(i+1)})$, and $C^{n(i+1)} \in \mathcal{C}_{n(i+1)}$.

By the above computation, $f_{n(i+1),n(i)}(V^{n(i+1)}) = H_{n(i)} \circ P(V^{n(i+1)})$. But the construction of $W_{n(i+1)}$ insures that $P(V^{n(i+1)}) = \hat{C}^{n(i+1)}$. The inductive hypotheses insure that there is some $C^{n(i)} \in \mathcal{C}_{n(i)}$ such that $C^{n(i+1)} \subset C^{n(i)}$, and hence $\hat{C}^{n(i+1)} \subset \hat{C}^{n(i)}$. Thus $H_{n(i)}(\hat{C}^{n(i)})$ is an element of $\mathcal{V}_{n(i)}$ containing $f_{n(i+1),n(i)}(V^{n(i+1)})$, and we see that

$$U^{n(i+1)} = \pi^{-1}_{n(i+1)}(V^{n(i+1)}) \subset \pi^{-1}_{n(i)}(H_{n(i)}(\hat{C}^{n(i)})).$$

Since $\pi^{-1}_{n(i)}(H_{n(i)}(\hat{C}^{n(i)})) \in \mathcal{U}_{n(i)}$, we have shown that $\mathcal{U}_{n(i+1)}$ refines $\mathcal{U}_{n(i)}$.

Next, we show that $\mathcal{U}_{n(i+1)}$ is "straight" in $\mathcal{U}_{n(i)}$. To make this more precise, for each $t \in \{n(1),n(2),\ldots\}$, let $U^t_1, U^t_2, \ldots, U^t_{g(t)}$ be the links of the chain $\mathcal{U}_t$, with a in U^t_1 and b in $U^t_{g(t)}$ (recall that $a = \pi^{-1}_n(x_0,0,0)$ and $b = \pi^{-1}_n(x_0,1,0)$ for every n), and $U^t_j \cap U^t_k \neq \emptyset$ if and only if $|j-k| \leq 1$. We need to show that if $U^{n(i+1)}_j \cup U^{n(i+1)}_k \subset U^{n(i)}_m$, where $j < k$, then for any integer p such that $j < p < k$, $U^{n(i+1)}_p \subset U^{n(i)}_m$.

Let $V^t_i \in \mathcal{V}_t$ and $C^t_i \in \mathcal{C}_t$ such that $U^t_i = \pi^{-1}_t(V^t_i)$ and $V^t_i = H_t(\hat{C}^t_i)$. Since $U^{n(i+1)}_j \cup U^{n(i+1)}_k \subset U^{n(i)}_m$, we have $f_{n(i+1),n(i)}(V^{n(i+1)}_j \cup V^{n(i+1)}_k) \subset V^{n(i)}_m$. But

$$f_{n(i+1),n(i)}(V_j^{n(i+1)} \cup V_k^{n(i+1)}) = H_{n(i)} \circ P(V_j^{n(i+1)} \cup V_k^{n(i+1)})$$

$= H_{n(i)}(\hat{C}_j^{n(i+1)} \cup \hat{C}_k^{n(i+1)})$, so that $H_{n(i)}(\hat{C}_j^{n(i+1)} \cup \hat{C}_k^{n(i+1)}) \subset H_{n(i)}(\hat{C}_m^{n(i)})$. Since $H_{n(i)}$ maps the interval $\{x_0\} \times I \times \{0\}$ homeomorphically onto the arc $\{(x,y,z) \in X_{n(i)}: x = x_0\}$, it follows that $H_{n(i)}(\hat{C}_p^{n(i+1)}) \subset H_{n(i)}(\hat{C}_m^{n(i)})$, and hence $U_p^{n(i+1)} \subset U_m^{n(i)}$.

Finally, we must show that $\lim_{i\to\infty} \mu(\mathcal{U}_{n(i)}) = 0$, where $\mu(\mathcal{U}_{n(i)})$ denotes the mesh of $\mathcal{U}_{n(i)}$. Let $w = (w_n)_{n=1}^{\infty}$ and $w' = (w'_n)_{n=1}^{\infty}$ be points in $U^{n(i)} \in \mathcal{U}_{n(i)}$. The distance between w and w' is $\rho(w,w') = \sup_n\{\frac{1}{n} d(w_n,w'_n)\}$, and $d(w_n,w'_n) = \max\{|P_i(w_n) - P_i(w'_n)|: i = 1,2,3\}$. Now, $w_{n(i)}$ and $w'_{n(i)}$ are in $V^{n(i)}$, where $U^{n(i)} = \pi_{n(i)}^{-1}(V^{n(i)})$, so that $P_1(\omega_{n(i)} = P_1(\omega'_{n(i)}) = x_0$, $|P_3(\omega_{n(i)} - P_3(w'_{n(i)})| < 2\varepsilon_{n(i)} < \lambda_{n(i)} < \frac{1}{2^{n(i)}}$, and diam $P_2(V^{n(i)}) = 2\beta_{n(i)} < \frac{1}{2^{n(i)}}$. Thus $d(w_{n(i)},w'_{n(i)}) < \frac{1}{2^{n(i)}}$. If $n > n(i)$, then $\frac{1}{n} d(w_n,w'_n) < \frac{1}{n} < \frac{1}{2^{n(i)}}$. If $n < n(i)$ then $d(w_n,w'_n) = d(f_{n(i),n}(w_n), f_{n(i),n}(w'_n)) < \frac{1}{2^{n(i)}}$ by Property 1 of the induction. Thus $\mu(\mathcal{U}_{n(i)}) < \frac{1}{2^{n(i)}}$, and hence $f^{-1}(x_0)$ is an arc.

CHAPTER III

MONOTONE COMPLETELY REGULAR MAPS

If C is a compact ANR and $\varepsilon > 0$, we will say that the number ε satisfies property (*) for C provided that

(*) if Z is any space and θ and ϕ are two maps of Z into C such that $d(\phi, \theta) < \varepsilon$, then θ and ϕ are homotopic.

Note that such an ε always exist (see, e.g., Hu [6]).

In this chapter the letter D will always denote the subspace of 3-dimensional Euclidean space E^3 given by $D = \{(x,y,z) \in E^3 \colon 0 \leq x \leq 1, -1 \leq y \leq 1, 0 \leq z \leq 1\}$. We will think of I^3 as the "right half" of D.

Lemma 3.1. Let A and B be disjoint closed subsets of a 1-dimensional compactum X. If A is connected and A is not cell-like, then there is a compact 3-manifold M and an embedding $\alpha\colon X \to M$ such that $\alpha(B)$ is contractible in M and $\alpha(A)$ is not contractible in M.

Proof. Embed A in the unit cube I^3 so that $A = \bigcap_{n=1}^{\infty} C_n$, where each C_n is a polyhedral cube with handles and $C_{n+1} \subset \operatorname{Int} C_n$ for each n. Since A is not cell-like there is an integer n such that A is not contractible in C_n.

Let k be the number of handles of C_n and let T be the standard solid torus with k holes in I^3. For each hole there is a line parallel to the z-axis which "goes through the hole." Choose one such line for each hole and denote these lines by $L_1, L_2, \ldots, L_k$. For each $i \in \{1, 2, \ldots, k\}$ let N_i be an open tubular neighborhood of L_i such that $N_i \cap N_j = \emptyset$ if $i \neq j$ and $N_i \cap T = \emptyset$ for all i. Let $M = D - (\bigcup_{i=1}^{k} N_i)$.

Let h be a homeomorphism of C_n onto T. Choose a point $p \in \operatorname{Int}(D - I^3)$ and define a mapping $g\colon A \cup B \to D$ by

$$g(x) = \begin{cases} h(x) & \text{if } x \in A \\ p & \text{if } x \in B. \end{cases}$$

Since X is 1-dimensional, g has an extension $\hat{g}\colon X \to \operatorname{Int} M$.

Let $\varepsilon > 0$ be a number which satisfies property (*) for M. By Lemma 1 of Wilson [22] there is an embedding α of X into Int M such that $d(\alpha,\hat{g}) < \min\{\varepsilon, d(p,I^3)\}$. Since $\hat{g}(A)$ is not contractible in M and $\hat{g}|_A$ is homotopic to $\alpha|_A$, $\alpha(A)$ is not contractible in M. Since $d(\alpha,\hat{g}) < d(p,I^3)$, $\alpha(B)$ is contractible in M.

Theorem 3.2. Let X be a 1-dimensional continuum and Y a nondegenerate continuum. If f is a monotone completely regular mapping of X onto Y, then f is cell-like.

Proof. Suppose f is not cell-like. Let y_1 and y_2 be two points in Y and let $A = f^{-1}(y_1)$ and $B = f^{-1}(y_2)$. Since A is connected, we may apply Lemma 3.1 to obtain a compact

3-manifold M and an embedding $\alpha: X \to M$ such that $\alpha(A)$ is not contractible in M and $\alpha(B)$ is contractible in M.

Let $Y_1 = \{y \in Y: \alpha(f^{-1}(y))$ is contractible in $M\}$ and $Y_2 = \{y \in Y: \alpha(f^{-1}(y))$ is contractible in $M\}$. The sets Y_1 and Y_2 are nonempty and disjoint, and their union is Y. We will obtain a contradiction to the fact that Y is connected by showing that Y_1 and Y_2 are both open.

Since M is a compact ANR, there is a number $\varepsilon > 0$ which satisfies property (*) for M. Let $\delta > 0$ be chosen so that if x_1 and x_2 are points in X with $d(x_1,x_2) < \delta$ then $d(\alpha(x_1),\alpha(x_2)) < \varepsilon$.

Let $y \in Y$. Since f is completely regular, there is an open set $V \subset Y$ with $y \in Y$ such that for any $z \in V$ there is a homeomorphism h of $f^{-1}(y)$ onto $f^{-1}(z)$ with $d(x,h(x)) < \delta$ for all $x \in f^{-1}(y)$. Thus $\alpha|_{f^{-1}(y)}$ and $(\alpha \circ h)|_{f^{-1}(y)}$ are ε-close and hence homotopic. Therefore, $\alpha(f^{-1}(y))$ is contractible in M if and only if $\alpha(h(f^{-1}(y))) = \alpha(f^{-1}(z))$ is contractible in M. Thus, for $i = 1$ or 2, $y \in Y_i$ if and only if $z \in Y_i$ for all $z \in V$. Therefore, Y_1 and Y_2 are open, a contradiction to the assumption that Y is connected. Thus, f must be cell-like.

Corollary 3.3. If X, Y and f are as in Theorem 3.2, then

(a) $\dim Y = 1$;

(b) for each $y \in Y$, $f^{-1}(y)$ is tree-like;

(c) if X is an ANR, then Y is an ANR;

(d) the map f is an hereditary shape equivalence.

Proof. Kozlowski and Walsh have shown in reference 12 that if X is a subset of a 3-manifold and $f: X \to Y$ is cell-like, then $\dim Y \leq \dim X$. Since, in our case $\dim X = 1$ and $\dim Y > 0$, we have $\dim Y = 1$. Properties (c) and (d) follow as a corollary to this same theorem. Property (b) is a result of the well-known fact that a 1-dimensional continuum is cell-like if and only if it is tree-like.

REFERENCES

1. R. D. Anderson, A continuous curve admitting monotone open maps onto all locally connected metric continua, (abstract), Bull. Amer. Math. Soc. 62(1956), 264.

2. V. Boltyanskii, An example of a two-dimensional compactum whose topological square is three-dimensional, Amer. Math. Soc. Transl. No. 48 (1951), 3-6; [Dokl. Akad. Nauk SSSR 67(1949), 597-599].

3. Haskell Cohen, A cohomological definition of dimension for locally compact Hausdorff spaces, Duke J. Math. 21(1954), 209-224.

4. Eldon Dyer, Certain transformations which lower dimension, Ann. of Math. 63(1956), 15-19.

5. I. Fáry, Dimension of the square of a space, Bull. Amer. Math. Soc. 67(1961), 135-137.

6. S. T. Hu, Theory of Retracts, Wayne State University Press, Detroit, Mich., 1965.

7. W. Hurewicz, Über oberhalb-stetige Zerlegungen von Punktmengen in Kontinua, Fund. Math. 15(1930), 57-60.

8. W. Hurewicz and H. Wallman, Dimension Theory, Princeton University Press, Princeton, N. J., 1948.

9. F. B. Jones, On a certain type of homogeneous plane continuum, Proc. Amer. Math. Soc. 6(1955), 735-740.

10. Y. M. Kazdan, An example of an open mapping of a one-dimensional locally connected continuum onto a square, Doklady Akad. Nauk. SSSR. N.S. 56(1947), 339-342.

11. A. Kolmogoroff, Über offene Abbildungen, Ann. of Math (2) 38(1937), 36-38.

12. G. Kozlowski and J. J. Walsh, The finite dimensionality of cell-like images of 3-manifolds, preprint.

13. J. Krasinkewicz and P. Minc, Continuous monotone decompositions of planar curves, preprint.

14. R. C. Lacher, Cell-like mappings, I, Pacific J. Math. 30(1969), 717-731.

15. ____________, Cell-like spaces, Proc. Amer. Math. Soc. 20(1969), 598-602.

16. A. Mason and D. C. Wilson, Monotone mappings on n-dimensional continua, Houston J. Math., to appear.

17. S. Mazurkiewicz, Sur les images continues des continus, Comptes Rendus du Premier Congrès des Mathematiciens des Pays Slaves, Warsaw, 1929-1930, 66-67.

18. L. Pontrjagin, Sur un hypothèse fondamental de la théorie de la dimension, C. R. Acad. Sci. Paris 130(1930), 1405-1407.

19. J. T. Rogers, Jr., Completely regular mappings and homogeneous, aposyndetic continua, preprint.

20. John J. Walsh, Dimension, cohomological dimension and cell-like mappings, preprint.

21. R. F. Williams, A useful functor and three famous examples in topology, Trans. Amer. Math. Soc. 106(1963), 319-329.

22. D. C. Wilson, Completely regular mappings and dimension, Bull. Amer. Math. Soc. 76(1970), 1057-1061.

23. ____________, Open mappings of the universal curve onto continuous curves, Trans. Amer. Math. Soc. 168(1972), 497-515.

BIOGRAPHICAL SKETCH

Dorothy Alice Mason was born on May 6, 1948, in Brooksville, Florida. She attended the Brooksville public schools and was graduated from Hernando High School in 1966. She attended Randolph-Macon Woman's College in Lynchburg, Virginia, from 1966 through 1968, and then transferred to the University of Florida, where she received the Bachelor of Arts degree, with a major in mathematics, in 1970. For the next five years she taught mathematics at High Point High School in Beltsville, Maryland. In 1975 she returned to the University of Florida as a graduate student in mathematics. She received the Master of Science degree in 1977. She spent the 1979-1980 academic year as a visiting student at the University of Tennessee in Knoxville, Tennessee. Since 1980, she has been a graduate student at the University of Florida.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

David C. Wilson, Chairman
Associate Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

James E. Keesling
James E. Keesling
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Mark L. Teply
Mark L. Teply
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Patrick J. McKenna
Associate Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Thomas E. Bullock
Thomas E. Bullock
Professor of Electrical Engineering

This dissertation was submitted to the Graduate Faculty of the Department of Mathematics in the College of Liberal Arts and Sciences and to the Graduate School, and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

April 1983

Dean for Graduate Studies
and Research